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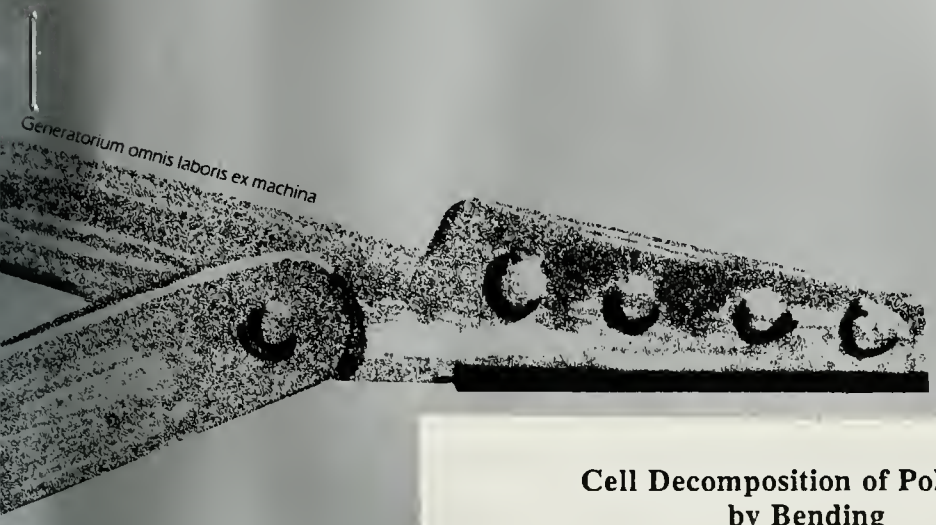

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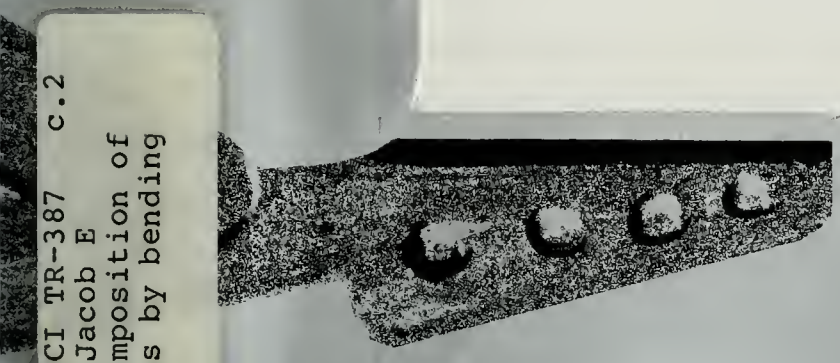

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Cell decomposition of polytopes by bending

Jacob E. Goodman*
János Pach**

1. Introduction.

Suppose a convex d-polytope P is cut by a hyperplane H not passing through any vertex of P. A recent result of Hovanski [6], also discovered independently by K. Bezdek, T. Bisztriczky, and R. Connelly [2], asserts that P must then have an uncut face of dimension at least $[\frac{d}{2}]$. Both proofs can be easily generalized to establish that P must have a pair of faces lying on opposite sides of H with total dimension at least $d-1$, as was suggested by the second named author.

Our main purpose here is to prove the following stronger statement:

Theorem 1. *Let H be a hyperplane crossing a d-dimensional convex polytope P. Assume that H does not contain any vertex of P, and let H^+ and H^- be the two halfspaces determined by H. Then P admits a cell decomposition whose maximal cells are openly disjoint convex polytopes $Q_1, Q_2, \ldots$ such that each $Q_i = \mathrm{conv}\{F_i \cup G_i\}$, where (i) $F_i \subset H^+$, $G_i \subset H^-$ are proper faces of P and (ii) $\dim F_i + \dim G_i = d-1$.*

In particular, if P is a simplicial polytope, then the resulting decomposition is a *triangulation*, i.e., each Q_i is a simplex.

The proof, which is presented in Section 2, uses a quite simple but apparently new method of cell decomposition. This construction, which may be thought of as "bending a polytope about a hyperplane", is related to the classical notion of "pulling vertices"[5] (cf. also [3] and [10]), yet is distinct from it. It consists essentially of projecting the vertices of the polytope P onto a wedge in R^{d+1} formed by two d-dimensional halfspaces meeting

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along H, taking the convex hull P' of the new set of vertices, and projecting the facets of the "upper" boundary of P' back into the original d-space. It is not hard to show that this results in a cell decomposition of P which is, in an obvious sense, transverse to H.

In Section 3 we consider another problem which can be resolved using the same method. It is well-known that, while every (not necessarily convex) planar polygon can be triangulated without introducing new vertices, the corresponding fact is not true in higher dimensions [8]. The same construction which proves Theorem 1 also enables us to give an affirmative answer to the following question posed recently by H. Edelsbrunner [4]: Can the region lying between two disjoint convex polytopes in d-space be triangulated without using additional vertices? More precisely, we prove

THEOREM 2. *Let P^1 and P^2 be two disjoint d-dimensional convex polytopes, and let P denote the (not necessarily convex) polytope obtained from $\mathrm{conv}\{P^1 \cup P^2\}$ by deleting P^1 and P^2 and then forming the closure. Then P admits a cell decomposition whose maximal cells are openly disjoint convex polytopes $Q_1, Q_2, \ldots$ such that each $Q_i = \mathrm{conv}\{F_i^1 \cup F_i^2\}$, where F_i^1 and F_i^2 are proper faces of P^1 and P^2, respectively, and $\dim F_i^1 + \dim F_i^2 = d-1$.*

Since each Q_i can clearly be triangulated from any of its vertices, Theorem 2 immediately implies that P has a simplicial decomposition with no extra vertices.

A slight variation of the same method gives

THEOREM 3. *Let P^1 and P^2 be two d-dimensional convex polytopes, and assume that P^1 contains P^2 in its interior. Let $P = P^1 \setminus \mathrm{int}(P^2)$. Then P admits a cell decomposition whose maximal cells are openly disjoint convex polytopes $Q_1, Q_2, \ldots$ such that each $Q_i = \mathrm{conv}\{F_i^1 \cup F_i^2\}$, where F_i^1 and F_i^2 are proper faces of P^1 and P^2, respectively, and $\dim F_i^1 + \dim F_i^2 = d-1$.*

In particular, P can be triangulated without introducing new vertices in such a way that each simplex of the decomposition contains vertices belonging to both P^1 and P^2.

Finally, in Section 4, we present a generalization of Theorem 1 to the situation in which

the vertices of P are partitioned into an arbitrary number of families by cones with common vertex in the interior of P. The construction which enables us to prove Theorem 1 must now be generalized somewhat, but we show that an analogous result holds.

At a meeting on discrete geometry (Oberwolfach, 1987), the second named author proposed the following conjecture (weaker than our Theorem 1, but stronger than the results of [2] and [6] mentioned at the beginning of the paper): Let H, H^+, H^-, and P be as in Theorem 1. Then, given any point $x \in P$, one can find two proper faces of P, $F \subset H^+$ and $G \subset H^-$, such that $x \in \text{conv}\{F \cup G\}$. Recently several independent arguments have been found to verify this conjecture. In fact, K. Bezdek has a proof in dimension 3 which apparently goes through in higher dimensions [1], J. Kincses has a beautiful topological proof using the Brouwer fixed point theorem [7], and the authors have also found a simple argument which relies on the Borsuk-Ulam theorem. All of these methods are purely existential, however, and seem inadequate to establish the stronger assertion of Theorem 1.

2. Proof of Theorem 1.

Let $x_1, \ldots, x_n$ denote the vertices of P. We choose the origin to be in H, and assume that

$$H^+ = \{x \in \mathbb{R}^d : \langle x, v \rangle > 0\}, \quad H^- = \{x \in \mathbb{R}^d : \langle x, v \rangle < 0\}$$

for some $v \in \mathbb{R}^d$. Suppose further that $x_1, \ldots, x_k \in H^+$ and $x_{k+1}, \ldots, x_n \in H^-$.

For any i, $1 \leq i \leq n$, let $x_i' \in \mathbb{R}^{d+1}$ be defined as follows:

$$x_i' = \begin{cases} (x_i, \langle x_i, v \rangle) & \text{if } i \leq k, \\ (x_i, 0) & \text{if } i > k. \end{cases} \tag{1}$$

Let $P' = \text{conv}\{x_i' : 1 \leq i \leq n\}$.

Notice that $\dim P'$ cannot be smaller than d, for the orthogonal projection of P' into $\mathbb{R}^d$ (the subspace induced by the first d coordinates) is equal to P.

We call a face Q' of P' an *upper facet*, if $\dim Q' = d$ and Q' can be "seen" from the positive "end" of the $(d+1)$st coordinate axis, i.e., if there exist $(s, t) \in \mathbb{R}^{d+1}$ ($s \in \mathbb{R}^d$,

$t > 0$) and a real α such that

$$\text{(2)} \qquad \begin{aligned} \operatorname{aff} Q' &= \{(x,y) \in \mathbb{R}^{d+1} : \langle x, s\rangle + yt = \alpha\} \\ P' &\subset \{(x,y) \in \mathbb{R}^{d+1} : \langle x, s\rangle + yt \leq \alpha\}. \end{aligned}$$

(Here aff Q' denotes the affine subspace of $\mathbb{R}^{d+1}$ spanned by the points of Q'.) Notice that in the special case when $\dim P' = d$, P' itself can, by abuse of language, be viewed as one single "upper facet".

It is easy to see that every upper facet Q' contains at least one vertex x_i' with $i \leq k$ and another with $i > k$. For if Q' did not contain any x_i' with $i \leq k$, then, in view of (1) and (2),

$$\operatorname{aff} Q' = \mathbb{R}^d = \{(x,y) \in \mathbb{R}^{d+1} : y = 0\}$$

and $\langle x_i, v\rangle \leq 0$ would hold for every $i \leq k$, contradicting our assumptions. Similarly, if Q' did not contain any x_i with $i > k$, then

$$\operatorname{aff} Q' = \{(x,y) \in \mathbb{R}^{d+1} : \langle x, -v\rangle + y = 0\}$$

and

$$\langle x_i, -v\rangle \leq 0 \text{ for every } i > k,$$

also a contradiction.

Let Q' be an upper facet of P', and let Q denote the orthogonal projection of Q' into $\mathbb{R}^d$. Set

$$I^+ = \{i : 1 \leq i \leq k \text{ and } x_i' \in Q'\},$$

$$I^- = \{i : k < i \leq n \text{ and } x_i' \in Q'\}$$

We claim that

$$\text{(3)} \qquad F = \operatorname{conv}\{x_i : i \in I^+\} \text{ and } G = \operatorname{conv}\{x_i : i \in I^-\} \text{ are proper faces of } P.$$

¿From this, and the fact that $Q = \operatorname{conv}\{F \cup G\}$, assertion (i) of the theorem will follow immediately, since P is clearly equal to the union of the orthogonal projections of the

upper facets of P' into $\mathbb{R}^d$, and no two of these projections have interior points in common; the fact that the induced decomposition of P is a legitimate cell decomposition, i.e., that the intersection of any two cells is again a cell, follows immediately from the corresponding fact about the boundary of P'.

To establish (3), observe that $H_G = (\operatorname{aff} Q') \cap \mathbb{R}^d$ is a hyperplane in $\mathbb{R}^d$ supporting Q, with the property that $H_G \cap Q = G$. On the other hand, let

$$H_F = \{x \in \mathbb{R}^d : \langle x, s + tv \rangle = \alpha\}.$$

Whenever $x_i' = (x_i, \langle x_i, v \rangle) \in Q'$ for some $i \leq k$, then, by (2),

$$\langle x_i, s \rangle + \langle x_i, v \rangle \, t = \alpha,$$

i.e., $x_i \in H_F$. If $x_i' = (x_i, 0) \in Q'$ for some $i > k$, i.e., $i \in I^-$, then by (2)

$$\langle x_i, s \rangle = \alpha,$$

so that $\langle x_i, s + tv \rangle < \alpha$. In other words, H_F is a supporting hyperplane of Q with $H_F \cap Q = F$. This completes the proof of (3), and hence of assertion (i) of the theorem.

To establish assertion (ii), let L^+ be the affine subspace of $\mathbb{R}^d$ spanned by the vertices of F, i.e., $L^+ = \operatorname{aff}\{x_i : i \in I^+\}$. Similarly, let $L^- = \operatorname{aff}\{x_i : i \in I^-\}$, and put $d^+ = \dim L^+ = \dim F$, $d^- = \dim L^- = \dim G$. Evidently, $d^+ + d^- \geq d - 1$ since Q is d-dimensional. Further, let $L_\circ^+$ and $L_\circ^-$ denote the translates of L^+ and L^-, respectively, passing through the origin.

If $\dim(L_\circ^+ \cap L_\circ^-) > 0$, then there exists $0 \neq u \in L_\circ^+ \cap L_\circ^-$ such that

$$u = \sum_{i \in I^+} \beta_i x_i = \sum_{i \in I^-} \beta_i (-x_i)$$

with suitable coefficients β_i satisfying $\sum_{i \in I^+} \beta_i = \sum_{i \in I^-} \beta_i = 0$. We can assume without loss of generality that $u \notin H$; otherwise we change the position of H slightly. But this

yields that $\dim Q' = d+1$, a contradiction. Indeed, given any point $(x,y) \in \mathsf{R}^{d+1}$ ($x \in \mathsf{R}^d$, $y \in \mathsf{R}$), one can find coefficients λ_i, $\sum_{i \in I^+ \cup I^-} \lambda_i = 1$, such that $x = \sum_{i \in I^+ \cup I^-} \lambda_i x_i$, thus

$$(x,y) = \sum_{i \in I^+ \cup I^-} \left(\lambda_i + \frac{y - \sum_{j \in I^+} \lambda_j \langle x_j, v \rangle}{\langle u, v \rangle} \beta_i \right) x_i', \tag{4}$$

i.e., (x,y) is in the affine span of Q'.

Hence $\dim(L_\circ^+ \cap L_\circ^-) = 0$, which implies that $d^+ + d^- \leq d$. We claim that

$$d^+ + d^- = d - 1. \tag{5}$$

Assume, in order to obtain a contradiction, that $d^+ + d^- = d$. Let $-L^-$ denote the reflection of L^- about the origin. Then $L^+ \cap (-L^-) \neq \emptyset$, hence there exists $u \in L^+ \cap (-L^-)$ such that

$$u = \sum_{i \in I^+} \beta_i x_i = \sum_{i \in I^-} \beta_i (-x_i)$$

for some β_i satisfying $\sum_{i \in I^+} \beta_i = \sum_{i \in I^-} \beta_i = 1$. We can again assume without loss of generality that $u \notin H$, and obtain a contradiction by (4), as above. This concludes the proof. □

3. Triangulating the space between two polytopes.

Rather than going through the proofs of Theorems 2 and 3 in detail, which would repeat much of the argument used in the proof of Theorem 1, we indicate only what modifications have to be made in each case.

PROOF OF THEOREM 2: Let H be a hyperplane strictly separating P^1 from P^2, let $x_1, \ldots, x_n$ denote the vertices of $P^1 \cup P^2$, and let H^+, H^-, $x_1', \ldots, x_n'$ and P' be as in the proof of Theorem 1. This time we want to project the *lower* facets of P' into R^d in order to obtain our cell decomposition. Thus, if we replace every reference to "upper" facet by "lower" facet in the proof of Theorem 1, and change $t > 0$ to $t < 0$ in (2), the argument then goes through *mutatis mutandis*. □

In order to prove Theorem 3, we modify the "bending" construction slightly:

PROOF OF THEOREM 3: Here we define the "wedge" in the proof of Theorem 1 so that it consists of two d-spaces which meet at infinity; i.e., we lift the polytope P^2 out of the d-space containing P^1 and into a parallel d-space. To this end, let $x_1, \ldots, x_k$ be the vertices of P^1 and $x_{k+1}, \ldots, x_n$ those of P^2, and define $x'_1, \ldots, x'_n \in \mathbb{R}^{d+1}$ by

$$x_i = \begin{cases} (x_i, 0) \text{ if } i \leq k, \\ (x_i, 1) \text{ if } i > k. \end{cases}$$

Let $P' = \text{conv}\{x'_i : 1 \leq i \leq n\}$. The facets of P' are now P^1 and (the lifted) P^2 themselves, plus others which project in $\mathbb{R}^d$ to a decomposition of P. Let Q'_i be one of the latter, and Q_i its orthogonal projection into $\mathbb{R}^d$. Now the assertion that Q_i is the join of faces of P^1 and P^2 is clear, since Q'_i is the span of a face of P^1 and a face of the lifted P^2. The only thing to worry about is the condition that $\dim F_i^1 + \dim F_i^2 = d - 1$, and this is achieved by an infinitesimal rotation of P^2, if necessary, which gets rid of any "partial parallelism" between the faces of P^1 and those of P^2. □

Notice that by combining Theorems 2 and 3, or alternatively by the method of proof of Theorem 4 below, Theorem 3 can be extended to yield a cell decomposition of the same type for a polytope P from which several other polytopes, $P_1, \ldots, P_k$ lying in its interior, each P_i separated by a hyperplane from the preceding ones, have been removed. We do not know whether the corresponding result holds without the separation condition.

4. Further generalizations.

Let Γ be a polytope containing O in its interior, and let $\partial\Gamma$ denote its boundary. Then Γ gives rise to a family $C^1, \ldots, C^k$ of convex cones partitioning $\mathbb{R}^d$ and to a convex function $f : \mathbb{R}^d \to \mathbb{R}$ whose restriction to each cone is a distinct linear function: let $C^i = \text{cone}_O(F_i)$ for each facet F_i of Γ, and let f be the linear extension to each cone of the function defined by $f(0) = 0$, $f(\partial\Gamma) = 1$. It is well-known, however, that not every decomposition of $\mathbb{R}^d$ into convex cones with a common vertex arises from a polytope in this manner [9]; let us call one which does *polytopal*.

Theorem 1 then admits the following generalization:

THEOREM 4. *Let P be a d-polytope containing the origin O in its interior, and let $C^1 \cup \cdots \cup C^k$, $f : \mathbb{R}^d \to \mathbb{R}$ be a polytopal decomposition of $\mathbb{R}^d$ such that each vertex of P lies in the interior of some C^j. Then P admits a cell decomposition $P = Q_1 \cup Q_2 \cup \ldots$ such that each $Q_i = \text{conv}\{F_i^1 \cup \cdots \cup F_i^k\}$, where (i) $F_i^j \subset C^j$ is a (possibly empty) face of P for each i, j, and (ii) $\sum_{1 \le j \le k} \dim F_i^j = d + 1 - k$.*

REMARK: Recall that, by convention, the empty face of a polytope has dimension -1. Hence notice that in the non-trivial case, when the vertices of P lie in more than one cone, say in $C_1, \ldots, C_m$, $m \ge 2$, it follows immediately that each Q_i has vertices belonging to at least two distinct cones, and (ii) can be strengthened to read $\sum_{1 \le j \le m} \dim F_i = d + 1 - m$.

PROOF: We follow the pattern of the proof of Theorem 1 insofar as possible. Let $x_1, \ldots, x_n$ be the vertices of P and, for each $i = 1, \ldots, n$, let

$$x_i' = (x_i, f(x_i)) \in \mathbb{R}^{d+1}. \tag{6}$$

Let $P' = \text{conv}\{x_1', \ldots, x_n'\}$ and let $I^1, \ldots, I^k$ be the sets into which $\{1, \ldots, n\}$ is partitioned by the condition

$$x_i \in C^j \text{ if and only if } i \in I^j.$$

Suppose, for each $j = 1, \ldots, k$, that the linear restriction of f to C^j is given by

$$f(x)|_{C^j} = \langle x, v_j \rangle .$$

Let Q' be an "upper facet" of P', i.e. a d-dimensional face such that

$$\begin{aligned} \text{aff } Q' &= \{(x, y) \in \mathbb{R}^{d+1} : \langle x, s \rangle + yt = \alpha\} \\ P' &\subset \{(x, y) \in \mathbb{R}^{d+1} : \langle x, s \rangle + yt \le \alpha\}. \end{aligned} \tag{7}$$

(As in the proof of Theorem 1, we admit the possibility that $Q' = P'$; this happens precisely when $\dim P' = d$.

Since the natural projection π from P' to P is surjective, it is clear that P is the union of d-cells of the form $Q_i = \pi(Q_i')$, with Q_i' an upper facet of P', and that no two of these

have interior points in common. Further, since all of these d-cells are convex, in order to show that they give rise to a legitimate cell decomposition of P it is sufficient to show that the intersection of any two such cells Q_i is a face of each, but this is immediate from the fact that the same holds for their preimages Q_i'; in fact π clearly induces a cell complex isomorphism between the upper boundary of P' and P with this decomposition.

It remains to show that each Q_i is a join of faces of the desired type.

Let Q' be an upper facet of P', and let $Q = \pi(Q')$. For $j = 1, \ldots, k$, let

$$I_Q^j = \{i : i \in I^j \text{ and } x_i' \in Q'\}.$$

We claim that

(8) $\qquad F^j = \operatorname{conv}\{x_i : i \in I_Q^j\}$ is a proper face of P for each j, $1 \leq j \leq k$.

To establish (8), let

$$H^j = \{x \in \mathbb{R}^d : \langle x, s + tv_j \rangle = \alpha\}.$$

Whenever $x_i' = (x_i, f(x_i)) \in Q'$ for some $i \in I^j$, then, by (7),

$$\langle x_i, s \rangle + \langle x_i, v_j \rangle t = \alpha,$$

i.e., $x_i \in H^j$. If, on the other hand, $i \notin I^j$, then, by (7), we still have

$$\langle x_i, s \rangle + f(x_i)t = \alpha;$$

but since f is convex and its restrictions to the cones C^j are all distinct, $\langle x, v_j \rangle < f(x_i)$, so that $\langle x_i, s \rangle + \langle x_i, v_j \rangle < \alpha$. In other words, H^j is a supporting hyperplane of Q with $H^j \cap Q = F^j$. This completes the proof of (8), and hence of assertion (i) of the theorem.

To establish assertion (ii), let L^j be the affine subspace of $\mathbb{R}^d$ spanned by the vertices of F^j for each j, i.e., $L^j = \operatorname{aff}\{x_i : i \in I^j\}$. Put $d_j = \dim L^j = \dim F^j$. Evidently, $\sum_{1 \leq j \leq k} d_j \geq d + 1 - k$ since Q is d-dimensional. (This implies, in particular, that Q

contains vertices belonging to at least two distinct cones C^j since, by (8), each d_j is at most $d-1$.) Further, let L_o^j denote the translate of L^j passing through the origin for each $j, 1 \leq j \leq k$.

Since any set I_Q^j which is empty contributes –1 to each side of (ii), it is sufficient to prove (ii) under the assumption that no $F_i^j = \emptyset$. For this purpose we perturb the cone system $C^1, \ldots, C^k$ slightly, if necessary, by perturbing the system of hyperplanes bounding Γ. (This may introduce new facets into Γ, or remove them, hence may alter the cone system, but only by adding or removing cones empty of vertices of P; it therefore has no effect on the validity of (ii).) In this way we can remove any dependence among the spaces L_o^j, so that—as in the proof of Theorem 1— the dimension statement follows. □

We get as an immediate corollary the following statement, which generalizes the result of [2] and [6] mentioned above:

COROLLARY. *If P, $C^1, \ldots, C^m$, and f are as in the Remark following Theorem 4, then some C^j contains a face of P of dimension at least $\frac{d+1-m}{m}$, and in particular an edge if $m \leq d$.*

We end with a question, whose positive answer would provide another generalization of the results mentioned at the end of Section 1.

Let K be a k-dimensional flat intersecting a d-polytope P. Assume that K does not contain any vertex of P. Then, given any point $x \in P$, can we find a flat L of dimension $d-k$ which passes through x and meets the boundary of P in only uncut faces?

As we have seen, the answer is "yes" for $k = d-1$. It is easy to see that it is also "yes" for $k = 1$.

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